(7B in Axler)

Lec 18 : Section 7.3 Convergence and Completeness

Setting: Normed linear space $(X, \|\cdot\|)$

Def: $f_n \to f$ in X in NORM, if

$$\|f_n - f\| \to 0$$

Examples: $C[a,b]$: $f_n \to f$ in $\|\ \|_{max}$ iff $f_n \to f$ uniformly

$L^\infty[E]$ $f_n \to f$ in $\|\ \|_\infty$ iff $f_n \to f$ a.e uniformly.

$L^p(E)$ $f_n \to f$ in L^p iff $\int |f_n - f|^p \to 0$

Cauchy sequences Recall that if $\{a_n\}$ is Cauchy

Then for any $\epsilon > 0$ $\exists$ N st $|a_n - a_m| < \epsilon$ $\forall n,m > N$

$\exists a$ st $a_n \to a$ iff $\{a_n\}$ is Cauchy.

$\Rightarrow$ $|a_n - a_m| \leq |a_n - a| + |a_m - a|$. —— (#1)

$\Leftarrow$ Pick a subsequence so that $|a_{n_k} - a_{n_{k+1}}| < \frac{1}{2^k}$

Cut $\sum_{k=1}^{m} a_{n_{k+1}} - a_{n_k}$ is absolutely convergent and so

$a = \lim_{m \to \infty} \sum_{k=1}^{m} a_{n_{k+1}} - a_{n_k} + a_{n_1}$. But

$$|a - a_n| \leq |a_n - a_{n_k}| + \sum_{r=k+1}^{\infty} |a_{n_r}| < 2\epsilon \quad (\#2)$$

large enough n and k.

Def: A sequence $\{f_n\}$ in a $(X, \|\cdot\|)$ is Cauchy if

for $\epsilon > 0$, $\exists N$ st $\|f_n - f_m\| < \epsilon$ $\forall n,m$

Completeness: A space X is complete if every $\{f_n\}$ Cauchy in X converge to some $f \in X$.

Such a space is called a Banach space

Ex: $\mathbb{R}$ with norm $|\ |$. We've proved that every Cauchy sequence is convergent so $\mathbb{R}$ is complete.

Ex: $C[a,b]$ is complete with max norm. $L^{\infty}(E)$ is complete using the same argument. (HW)

Prop 4: If $f_n \to f$ in X then f_n is Cauchy.
If f_n is Cauchy and has a convergent subsequence then $f_n \to f$ in X.

Pf: If $f_n \to f$ in X $\{f_n\}$ is Cauchy from the triangle inequality. see (#1).

If f_n is Cauchy and $\{f_{n_k}\}_{k=1}^{\infty} \to f$ then $f_n \to f$.

The idea is similar to (#2).

Def: $\{f_n\}$ is rapidly Cauchy if $\exists \{\epsilon_k\}_k \in \mathbb{R}^+$ st
$\sum_{k=1}^{\infty} \epsilon_k < \infty$ and $\|f_{k+1} - f_k\| \leq \epsilon_k^2 \quad \forall k$.

Note: if $\{f_n\}$ is rapidly Cauchy

$$\|f_{n+k} - f_n\| \leq \sum_{i=n}^{n+k-1} \epsilon_k^2$$

Prop 6: Every rapidly Cauchy sequence is Cauchy and every Cauchy sequence has a rapidly Cauchy subsequence.

Pf: If $\{f_n\}$ is Rapidly Cauchy $\|f_{n+k} - f_n\| \le \sum_{j=n}^{n+k-1} \epsilon_k^2$

if $\sum_{k=1}^{\infty} \epsilon_k$ is convergent, $\epsilon_k \downarrow 0$ and hence

$\sum_{k=1}^{\infty} \epsilon_k^2$ is convergent. So for large enough n,

$$\|f_{n+k} - f_n\| \le \sum_{j=n}^{\infty} \epsilon_k^2 < \epsilon .$$

So $\{f_n\}$ is Cauchy.

As before, if f_n is Cauchy, choose n_k st

$\|f_n - f_m\| \le \frac{1}{2^{2k}}$ for $n, m > n_k$.

Then f_{n_k} is rapidly Cauchy with $\epsilon_k = \frac{1}{2^k}$

Theorem 6: Let E be a meas set $1 \le p \le \infty$. Then every rapidly Cauchy seq in L^p conv. in L^p and pointwise a.e.

Pf: Choose $\sum_{k=1}^{\infty} \epsilon_k$ st $\|f_{k+1} - f_k\|_p \le \epsilon_k^2$

and hence $\int |f_{k+1} - f_k|^p \le \epsilon_k^{2p}$

$$|f_{k+1}(x) - f_k(x)| > \epsilon_k \iff |f_{k+1}(x) - f_k|^{2p} > \epsilon_k^{2p}$$

~~and so $m(|f_{k+1} - f_k(x)| > \epsilon_k)$~~

$$= m\left(|f_{n+1} - f_n(x)|^p > \epsilon_n^p\right) \leq \frac{1}{\epsilon_n^p}\int |f_{n+1} - f_n|^p$$

$$\leq \frac{\epsilon_n^{2p}}{\epsilon_n^p} = \epsilon_n^p$$

~~Since~~ $p \geq 1$, $\sum \epsilon_n^p$ converges. The BC Lemma can be applied to

$$E_n = \{x \mid |f_{n+1} - f_n(x)| > \epsilon_n\}$$

since $\sum_k m(E_n) = \sum_k \epsilon_n^p < \infty$.

Then $m(E_n \text{ i.o}) = 0$ so for each $x \in \{E_n \text{ i.o}\}^c$,

$\exists$ a $K(x)$ st $|f_{n+1} - f_n(x)| < \epsilon_n^p \quad \forall\ k > k(x)$

Hence $|f_{k+m} - f_k(x)| < \sum_{m=k}^{\infty} \epsilon_n < \epsilon$

So for each $x \in \{E_k \text{ i.o}\}^c$, the sequence $\{f_n(x)\}$ is Cauchy and hence $\exists\ f$ st

$f_n(x) \longrightarrow f(x) \quad \forall\ x \in \{E_n \text{ i.o}\}^c$

Since $\{f_n\}$ is Cauchy in L^p

$$\int_E |f_{n+k} - f_k|^p \leq \left[\sum_{j=k}^{k+n} \epsilon_j^{2p}\right] \leq \sum_{j=k}^{\infty} \epsilon_j^{2p}$$

Since $f_k \longrightarrow f$ a.e on E by Fatou we can take a limit as $n \to \infty$ to get

$$\int_E |f - f_k|^p \leq \sum_{j=k}^{\infty} \epsilon_j^{2p}$$

$$\Leftrightarrow \quad \|f - f_k\|_p \longrightarrow 0$$

How to show that f is in L^p?

$$\|f\|_p \leq \|f - f_k\|_p + \|f_k\|_p < \infty$$

Riesz - Fischer Theorem: Let E be meas. and $1 \leq p \leq \infty$

$L^p(E)$ is a Banach Space and

moreover

$$f_n \to f \text{ in } L^p \Rightarrow f_n \to f \text{ in meas}$$

$$\Rightarrow \exists \{n_k\} \text{ st } f_{n_k} \to f \quad a.e$$

Pf: Let $\{f_n\}$ be Cauchy, then $\exists$ a subseq that is rapidly Cauchy $\Rightarrow \{f_{n_k}\} \to f$ in L^p. But of a Cauchy seq has a convergent subseq, this means $f_n \to f$ in L^p. L^p convergence implies convergence in meas. Therefore $\exists$ m_k st $f_{m_k} \to f$ a.e.

Example For $E = [0,1]$ $1 \leq p < \infty$ and each natural number n, let $f_n = n^{1/p} \chi_{(0,1/n]}$. $f_n \to 0$ everywhere. But $\|f_n\| \not\to 0$.

Thm: Let E meas and $1 \le p < \infty$. Suppose $\{f_n\} \in L^p$ and $f_n \to f$ a.e. and $f \in L^p$. Then

$$\{f_n\} \to f \text{ in } L^p \iff \int |f_n|^p \to \int |f|$$

Pf:
Minkowski says

$$\big|\, \|f_n\|_p - \|f\|_p \,\big| \le \|f_n - f\|_p$$

Hence $\{f_n\} \to f$ in L^p $\quad \int |f_n|^p \to \int |f|$. To prove the converse

$\varphi(t) = t^p$ is convex for $t \ge 0$ and $p \ge 1$, so

$$\varphi\left(\frac{a+b}{2}\right) \le \frac{\varphi(a) + \varphi(b)}{2}$$

$$0 \le \frac{|a|^p + |-b|^p}{2} - \frac{|a-b|^p}{2^p}$$

Then $h_n \ge 0$ where $h_n(x) = \dfrac{|f_n(x)|^p + |f(x)|^p}{2} - \dfrac{|f_n(x) - f(x)|^p}{2^p}$

Then by Fatou

$$\int \lim h_n \le \underline{\lim} \int \frac{|f_n|^p + |f(x)|^p}{2} - \frac{|f_n - f(x)|^p}{2^p}$$

$\lim h_n = |f(x)|$ by pointwise convergence

So rearranging,

$$\|f\|_p^p + \overline{\lim}\, \|f_n - f\|_p^p \le \|f\|_p^p$$

$$\Rightarrow \quad \|f_n - f\|_p \to 0$$

Thm 8 Let E be meas and $1 \leq p < \infty$.

Suppose $f_n \in L^p$ and $f_n \to f$ a.e. AND $f \in L^p$

Then

$$\{f_n\} \to f \text{ in } L^p$$

$$\Longleftrightarrow$$

$|f_n|^p$ is UI and tight over E.

Pf: as before $|f_n - f|^p \to 0$

$\Longleftrightarrow$ $|f_n - f|^p$ is UI and tight.

from Vitali. But $|f_n - f|^p$ is UI and tight

$\Longleftrightarrow$ $|f_n|^p$ is UI and tight

$|f_n - f|^p \leq 2^p(|f_n|^p + |f|^p)$ | gives $|f_n|^p$ UI and tight $\Rightarrow$ $|f_n - f|$ UI and tight

$|f_n|^p \leq 2^p(|f_n - f|^p + |f|^p)$

By assumption $f \in L^p$ so gives $|f_n|^p$ UI and tight